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AN ALGORITHM FOR THE DECOMPOSITION OF ARC FLOWS INTO PATH FLOWS FOR THE GSPEP

by

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This research was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

Cette étude a été publiée grâce à une subvention du fonds F.C.A.R. pour l'aide et le soutien à la recherche.

Université de Montréal
Centre de recherche sur les transports - Publication #569
April 1988

Dépôt légal - 2e trimestre 1988

Abstract

The General Spatial Price Equilibrium Problem (GSPEP) may be formulated either in terms of path flow variables or in terms of arc flow variables.

When the variables are the path flows, the solution leads to important informations that are the origin-destination demand matrix, and the paths used to transport goods. When the problem is solved in terms of the arc flows, the solution gives only the flows on the arcs of the network.

On the other hand, the algorithms designed in literature to solve the GSPEP in terms of arc flows are more efficient than those designed for the path flow formulation (this is generally due to the huge number of paths).

In this paper, we propose an algorithm, that we call CAFPF, that decomposes the arc flow equilibrium into a path flow equilibrium (by a sequence of minimal path cost generation). CAFPF is then used after that the GSPEP has been solved in terms of arc flows, leading to the information on the O/D demand matrix and a transportation strategy. Numerical results show the efficiency of CAFPF.

Finally we note that CAFPF can be used for the linear or nonlinear, symmetric or asymmetric cases of GSPEP, and that it can easily be generalized to other problems such the Network Equilibrium Problem.

Key words : GSPEP, NEP, arc flows, path flows.

Résumé

Le problème d'équilibre spatial sur un réseau général (PESG) peut être formulé soit en termes de flots sur les chemins, soit en termes de flots sur les arcs.

Lorsque les variables sont les flots sur les chemins, la solution donne des informations importantes, à savoir la matrice des flots entre les paires O/D et les chemins sur lesquels le bien est transporté. Lorsque le problème est résolu en termes de flots sur les arcs, la solution ne donne que les flots sur les arcs.

D'un autre côté, les algorithmes de résolution du PESG dans l'espace des flots d'arcs sont plus performants que ceux dans l'espace des chemins (en général à cause du nombre très élevé de chemins).

Dans cet article, nous proposons un algorithme, que l'on appelle CAFPF, qui décompose les flots d'arcs à l'équilibre en flots de chemins. CAFPF est alors utilisé après que le PESG ait été résolu en termes d'arcs, permettant ainsi d'obtenir les informations sur la matrice O/D des flots, et sur les chemins utilisés pour le transport du bien. Les résultats numériques ont démontré l'efficacité de CAFPF.

Finalement, nous aimerions souligner que CAFPF peut être utilisé pour les PESG linéaires ou non linéaires, symétriques ou asymétriques, et qu'il peut être facilement généralisé à d'autres problèmes tel le problème d'équilibre de réseau.

Mots-clés : PESG, PER, flots d'arcs, flots de chemins.

1- Introduction

In this paper, we consider the GSPEP (General Spatial Price Equilibrium Problem) which has been widely studied in literature ([3] to [10], [12], [13], [15], [16]); it is known that this problem can be formulated either in terms of path flow variables or in terms of arc flow variables.

Mathematically, the formulation may take one of the two following forms:

- Path flow formulation : $minf(h)\ s.t.\ h \geq 0$

- Arc flow formulation : $minf(v)\ s.t.\ Av = 0,\ v \geq 0$

In the first formulation, the vector h refers to the path flow variables $h_k,\ k \in K$ where the set K is the set of all the paths from all the origins i to all the destinations j in the network.

In the second formulation, v refers to the arc flow variables, and the constraint $Av = 0$ expresses flow conservation conditions where A corresponds to the node-arc incidence matrix.

In both formulations, f is the objective function to be minimized.

The formulation in terms of path flows seems more simple than the one in the space of arc flows since it does not contain flow conservation conditions. But when one decides to solve the problem in terms of path flows, one meets a major difficulty, that is the huge number of variables (number of paths in K).

On the other hand, the number of variables in the arc flow formulation is in the order of the number of arcs, number which is much more smaller than the number of paths. It is true that the formulation in terms of arc flows contains flow conservation conditions (one at each node), but this constraint is generally easier to handle than a huge number of variables.

For this reason, usually, the problem is solved in terms of arc flow variables either considering explicitly the constraint of flow conservation, or using some minimal cost path strategy that handles implicitly that constraint. Yet, in our knowledge, there is no algorithm in terms of arc flows that keeps the information on the used paths; so at equilibrium, the equilibrium flows on the arcs are known, but there is no information neither on a path strategy to transport goods from origins to destinations, nor on the origin-destination demand matrix.

These important informations are given by a path flow formulation, but, in our knowledge, there is only one algorithm ([3],[6]) that solves explicitly the problem in terms of path flows, but it remains less efficient than some algorithms proposed to solve the GSPEP in the arc flow space ([4],[6]).

In this paper, we propose an efficient algorithm that decomposes the equilibrium arc flows into equilibrium path flows. The interest of such an algorithm is the following : after having solved the problem, in terms of arc flows, by an efficient known algorithm, the algorithm that we call CAFPF (conversion of arc flows into path flows) will allow to obtain a path flow equilibrium, leading then to all the informations that the formulation in the arc flow space do not give (e.g. O/D demand matrix and transportation strategy).

Before to present the algorithm, we would like to mention that the path flow equilibrium is generally not unique, and even if the problem was solved explicitly in terms of path flows, the solution would be one path equilibrium among others. As a consequence, the solution obtained by CAFPF is a path flow equilibrium among others, but is after all an equilibrium.

In section 2 we present the proposed algorithm CAFPF and give a proof of convergence. In section 3, some numerical results are presented before the conclusion in section 4.

2- Algorithm CAFPF for GSPEP

In this section, we develop the algorithm CAFPF for the General Spatial Price Equilibrium Problem.

The equilibrium conditions of the GSPEP in terms of path flows are the following:

$$S_i(d_i) + \sum_{a \in k} C_a(v) - I_j(d_j) \begin{cases} = 0 & if\ h_k > 0 \\ \geq 0 & otherwise \end{cases} \quad \forall i \in O,\ \forall j \in D,\ \forall k \in K_{ij}$$

where:

O	is the set of origin nodes,
D	is the set of destination nodes,
K_{ij}	is the set of all paths from node i to node j,
h_k	is the flow on path k,
v	is the vector of flows on the arcs,
v_a	is the flow on arc a with $v_a = \sum_k \delta_{ak} h_k$
δ_{ak}	$= \begin{cases} 1 & if\ arc\ a\ is\ in\ path\ k \\ 0 & otherwise \end{cases}$

For convenience, we change the notation for ([2],[4],[6]) :

$$\begin{array}{rcl} S_i & \equiv & C_{mi} \quad and \quad d_i \equiv v_{mi} \quad \forall i \in O \\ I_j & \equiv & -C_{jm} \quad and \quad b_j \equiv v_{jm} \quad \forall j \in D \end{array}$$

This corresponds to (fictitiously) add one supernode m to the set of nodes, and arcs (m,i), $\forall i \in O$, and arcs (j,m), $\forall j \in D$, to the set of arcs. Under this view, the optimality conditions become in this "augmented" network:

$$\sum_{a \in k} C_a(v) \begin{cases} = 0 & if\ h_k > 0 \\ \geq 0 & otherwise \end{cases} \quad \forall k \in K$$

where K is the set of all paths from node m to node m in the augmented network.

It becomes obvious from these optimality conditions that a used path has a cost equal to the cost of the minimal cost path (with a zero cost), and a nonused path has a cost greater or equal to the minimal cost (here zero).

Now, since we suppose that the problem has been solved in terms of arc flows, the equilibrium vector of arc flows v is known, then so it is for the costs $C_a(v)$ for all the arcs (of the augmented network). Hence, it is possible to determine minimal cost paths (from m to m), and to affect on them the flow actually on the arcs of the network.

To visualize this idea, let's describe the algorithm in detail:

Algorithm CAFPF

$0 - Initialize : K^0 = \emptyset,\ n = 0,\ \forall a\ v_a^0 = v_a\ and, C_a^0 = \begin{cases} C_a(v) & if\ v_a^n > 0 \\ \infty & otherwise \end{cases}$

1- $While\ \exists a \mid v_a^n \neq 0\ do$

$i - Generate\ the\ minimal\ cost\ path\ k\ (using\ C_a^n{'s}),\ and$

$.....let\ K^{n+1} = K^n \bigcup \{k\};$

$.....if\ k \notin K^n\ then\ let\ h_k^n = 0.$

$ii - Let\ \nu^n = min\{v_a^n,\ a \in k\}$

$iii - \forall a\ do : v_a^{n+1} = \begin{cases} v_a^n - \nu^n & if\ a \in k \\ v_a^n & otherwise \end{cases}$

$.....and\ \forall k' \in K^{n+1}\ do : h_{k'}^{n+1} = \begin{cases} h_{k'}^n + \nu^n & if\ k' = k \\ h_{k'}^n & if\ k' \neq k \end{cases}$

$iv - \forall a\ do : C_a^{n+1} = \begin{cases} C_a^n & if\ v_a^n > 0 \\ \infty & otherwise \end{cases}$

$v - Let\ n = n + 1.$

Note : In the algorithm CAFPF, setting some arc costs at infinity is equivalent to removing these arcs of the network.

Theorem: *The algorithm CAFPF completes the decomposition of arc flows into path flows in a finite number of iterations (less or equal to the number of arcs).*

In order to prove the theorem, we state the two following lemmas:

Lemma 1 : *At any iteration, the solution v_a remains unchanged, so:*

$$v_a^n + \sum_{k' \in K^n} \delta_{ak'} h_{k'}^n = v_a\ \forall\ n,\ \forall\ a.$$

Note: This relation means that at each iteration, a part of the arc flows is decomposed into path flows.

Proof of lemma 1:

This relation is obviously true for n=0 since the set K^0 is empty. Now suppose that the relation is satisfied for n. We need to prove that it is satisfied for n+1 too.

- Consider first the arcs $a \notin k$ where k is the path generated at this iteration. Then :

$$\begin{aligned} v_a &= v_a^n + \sum_{k' \in K^n} \delta_{ak'} h_{k'}^n \\ &= v_a^{n+1} + \sum_{k' \in K^n} \delta_{ak'} h_{k'}^n && \text{because } a \notin k \text{ so } v_a^n = v_a^{n+1} \\ &= v_a^{n+1} + \sum_{k' \in K^{n+1}} \delta_{ak'} h_{k'}^n && \text{because } \delta_{ak} = 0, \text{ and } K^{n+1} = K^n \bigcup \{k\} \\ &= v_a^{n+1} + \sum_{k' \in K^{n+1}} \delta_{ak'} h_{k'}^{n+1} && \text{because } \delta_{ak} = 0, \text{ and } h_{k'}^n = h_{k'}^{n+1} \forall k' \neq k \end{aligned}$$

- Consider now arcs $a \in k$:

$v_a^{n+1} = v_a^n - \nu^n$ *and* $h_k^{n+1} = h_k^n + \nu^n$.

So:

$$\begin{aligned} v_a^{n+1} + \sum_{k' \in K^{n+1}} \delta_{ak'} h_{k'}^{n+1} &= v_a^n - \nu^n + \sum_{k' \in K^{n+1}, k' \neq k} \delta_{ak'} h_{k'}^{n+1} + h_k^{n+1} \\ &= v_a^n - \nu^n + \sum_{k' \in K^{n+1}, k' \neq k} \delta_{ak'} h_{k'}^n + h_k^{n+1} \end{aligned}$$

If $k \notin K^n$ *then* $\{k' \neq k, k' \in K^{n+1}\} = K^n$, *and* $h_{k'}^n = h_{k'}^{n+1} \forall k' \neq k$, *so* :

$$\begin{aligned} v_a^{n+1} + \sum_{k' \in K^{n+1}} \delta_{ak'} h_{k'}^{n+1} &= v_a^n - \nu^n + \sum_{k' \in K^n} \delta_{ak'} h_{k'}^n + 0 + \nu^n \\ &= v_a^n + \sum_{k' \in K^n} \delta_{ak'} h_{k'}^n = v_a \text{ by recursivity} \end{aligned}$$

otherwise $k \in K^n$, *so* $K^{n+1} = K^n$. *Then* :

$$\begin{aligned} v_a^{n+1} + \sum_{k' \in K^{n+1}} \delta_{ak'} h_{k'}^{n+1} &= v_a^n - \nu^n + \sum_{k' \in K^n, k' \neq k} \delta_{ak'} h_{k'}^n + h_k^n + \nu^n \\ &= v_a^n + \sum_{k' \in K^n} \delta_{ak'} h_{k'}^n = v_a \text{ by recursivity.} \end{aligned}$$

Lemma 2: *At each iteration n, if the arc flow v is not completely decomposed into path flows, then the minimal cost path k generated do not contain any arc a with a zero flow* v_a^n .

Proof of lemma 2:

Let k the minimal cost path generated at this iteration. In the algorithm, each arc a with $v_a^n = 0$ has a cost $C_a^n = \infty$. Suppose now, by contradiction, that the path k contains an arc a with $v_a^n = 0$, so the cost of the minimal cost path k is infinity. Then all the other paths have a cost equal to infinity. Since the only arcs a that have an infinite cost C_a^n are those with a flow $v_a^n = 0$, then any path contains an arc a with a $v_a^n = 0$, so by flow conservation, $\forall a \in k$, $\forall k \in K, v_a^n = 0$, which means (by lemma 1) that the arc flow v is completely decomposed into path flows, contradicting the hypothesis of the lemma.

Proof of the theorem:

Under the hypothesis that the arc flows are not completely decomposed into path flows, the minimal cost path k generated at iteration n do not contain any arc a with $v_a^n = 0$ (by lemma 2). Then ν^n is nonnull at step ii. So at step iii of the algorithm, at least one arc a with a $v_a^n \neq 0$ will have $v_a^{n+1} = 0$. Since the network is finite, the number of arcs is finite, so in a finite number of iterations, all the arcs will have a v_a^n null, meaning (by lemma 1) that the arc flows v have been completely decomposed into path flows.

3- Numerical Results

In the following, we present some numerical results of the algorithm CAFPF which has been coded in PASCAL on the CYBER 835 of the Université de Montréal.

Six networks have been randomly generated to test the efficiency of the algorithm CAFPF. The networks' size varies between 10 nodes/32 arcs and 100 nodes/697 arcs (usual sizes of GSPE problems).

On the other hand, we use the algorithms GSNPR ([3],[6]) and NAC ([4],[6], [12]) that represent, in our knowledge, the state of the art in solving GSPE problems.

The first algorithm, GSNPR, is a Gauss-Seidel-Newton-Projection type algorithm, with a strategy of restriction, that solves the GSPEP in terms of path flows. In our knowledge, GSNPR is the only algorithm that solves the GSPEP in terms of path flows; other usual optimization algorithms are impractical due to the huge number of variables (paths).

The second, NAC, is a descent algorithm; the descent direction is obtained by a Newton-Active-Constraints type algorithm where the initial active set is chosen using the Lagrange multipliers, and the active-constraints method is stopped as soon as a nonnull descent direction is obtained (more details are in [4],[6]).

In table 1, we produce results obtained by NAC, CAFPF and GSNPR on the CYBER 835 of the Université de Montréal.

Column "SIZE" gives the size of the test networks (generated randomly).

In the objective function, the arc cost functions used for those networks are nonlinear (cubic except for destinations where they are logarithmic).

Column "NAC" gives the CPU time (seconds) used to solve the GSPEP in terms of arc flows (by NAC).

Columns "CAFPF" gives the CPU time to decompose the arc flow equilibrium (obtained by NAC) into a path flow equilibrium, and the number of iterations used for the decomposition (this number corresponds to the number of paths in the path flow equilibrium obtained by CAFPF).

Column "NAC+CAFPF" gives the total of the CPU time used to solve the GSPEP in terms of arc flows by NAC plus the CPU time used to decompose the arc flow equilibrium into a path flow equilibrium (by CAFPF).

Column "GSNPR" gives the CPU time used by the algorithm GSNPR to solve the problem in terms of path flows. Compared with the previous column, it seems advantageous to solve the GSPEP by NAC and decompose the solution by CAFPF.

It can be concluded from these results that the algorithm CAFPF can be used efficiently, in combination with an algorithm that works in the arc flows space, to obtain a path flow equilibrium.

Table 1

SIZE	NAC	CAFPF		NAC+CAFPF	GSNPR
	Cpu	Cpu	Iter	Cpu	Cpu
10/32	0.66	0.11	13	0.77	0.91
20/112	2.15	0.82	49	2.97	6.54
30/260	5.71	3.05	104	8.76	19.71
40/272	6.45	4.70	113	11.15	25.65
50/435	10.57	12.10	193	22.67	43.86
100/697	38.25	67.00	305	105.25	135.43

4- Conclusion

In this paper, we consider the GSPEP (General Spatial Price Equilibrium Problem) which can be formulated either in terms of path flows or in terms of arc flows.

It is known that the algorithms designed in literature to solve the problem in terms of arc flows are more efficient than those designed to solve the problem in terms of path flows. Unfortunately, the formulation in terms of arc flows does not give the informations on the origin-destination demand matrix and on the strategy of transportation (paths).

In this paper, we propose an algorithm that we call CAFPF (for Conversion of Arc Flows into Path Flows) that decomposes the equilibrium arc flows into a path flow equilibrium; first the GSPEP is solved by an efficient algorithm that works in the arc flow space, then CAFPF is used to decompose the (equilibrium) arc flows into (equilibrium) path flows, leading to the information on the O/D demand matrix and a transportation strategy.

Numerical results show the efficiency of CAFPF on usual GSPEP network sizes.

Now, we would like to note that since CAFPF is used at equilibrium, it considers the equilibrium values of the transportation costs on the arcs, so it does not matter if the transportation cost functions used to solve the problem are linear or not, symmetric or not. So CAFPF can be used for symmetric or asymmetric, linear or nonlinear GSPEP.

Finally, we note that CAFPF can be easily generalized to other problems such the Network Equilibrium Problem (NEP) ([1], [14]) by simply using step 1 of CAFPF in a loop on O/D pairs and on commodities.

Acknowledgement

This paper benefited from a careful reading by Professor Michael Florian.

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